

## ASPECTS OF THE PARAMETRIC ANALYSIS OF HYPERSTATIC PLANE SYSTEMS COMPOSED OF THREE WELDED BEAMS (II)

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**Abstract:** The parametric analysis and design became of interest in the last time period due to the development of the computing technology and are widely used by the professional software. The parametric design is employed in the 2D and 3D professional modeling software, because it allows the reconstruction and change of the virtual model by changing one or more dimensional parameters. This work presents a simple example of parametric analysis of hyperstatic plane systems composed of three perpendicularly welded beams of the same length, using the force method for various loading scenarios with concentrated and distributed loads and bending moments.

**Keywords:** parametric analysis, hyperstatic (statically indeterminate) systems, force method.

### 1. INTRODUCTION

A plane hyperstatic frame is considered, composed of three perpendicularly welded beams of the same length  $L$ , fixed at the both ends, subjected to a loading scenario represented in the Fig. 1. The beams have a constant cross-section and are subjected to bending, shearing and/or tension/compression under the action of:

- the concentrated forces  $P_1$ ,  $P_2$  and  $P_3$  acting on the beams at the locations  $\alpha_P L$ ,  $\beta_P L$  and  $\gamma_P L$ ;
- the bending moments  $N_1$ ,  $N_2$  and  $N_3$  acting on the beams at the locations  $\alpha_N L$ ,  $\beta_N L$  and  $\gamma_N L$ ;
- the uniform distributed loads  $q_1$ ,  $q_2$  and  $q_3$  acting on the beam segments  $\mu L$ ,  $\theta L$  and  $\psi L$  respectively, at the locations  $\eta L$ ,  $\omega L$  and  $\tau L$ .

For the particular values of the parameters, given in the Table 1, the reaction forces in the supports A and B will be calculated, along with the rotation of the section C.

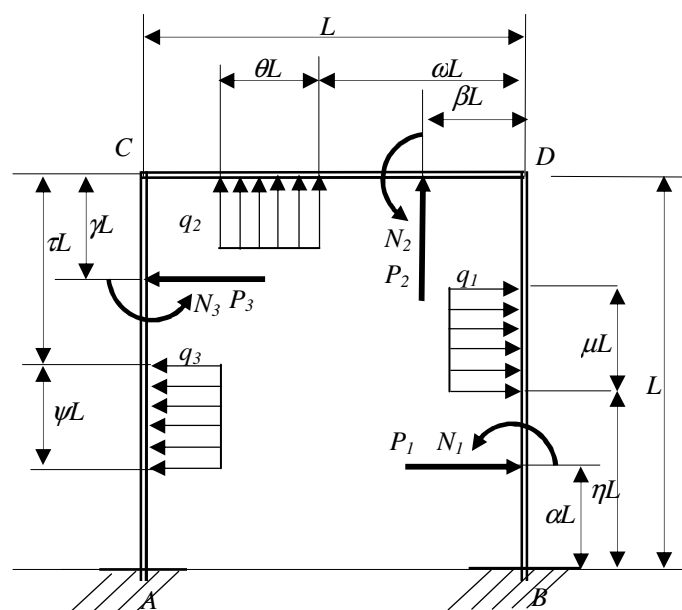
**Table 1. Particular values of the parameters.**

$N_1$	$PL$	$\alpha_N$	$1/2$
$N_2$	-	$\beta_N$	-
$N_3$	-	$\gamma_N$	-
$P_1$	-	$\alpha_P$	-
$P_2$	$-2P$	$\beta_P$	$1/2$
$P_3$	-	$\beta_P$	-
$q_1$	-	$\eta$	-
		$\mu$	-
$q_2$	-	$\omega$	-
		$\theta$	-
$q_3$	$-2P/L$	$\tau$	$0$
		$\psi$	$1/2$

The parametric analysis of hyperstatic frame is represented by the determination of the parametric expressions of displacements for the base system and by the numerical computations for different particular values of the parameters.

### 2. FORCE METHOD

The force method allows the determination of the hyperstatic unknowns by choosing a statically determined system (called base system), which is mechanically equivalent to the given hyperstatic system. The base system is obtained by replacing the fixed support in the point B with mechanical elements: the forces  $X_1$  and  $X_2$  and the moment  $X_3$  – which are the hyperstatic unknowns (Fig. 2).



**Fig. 1. Frame layout and loading scheme**

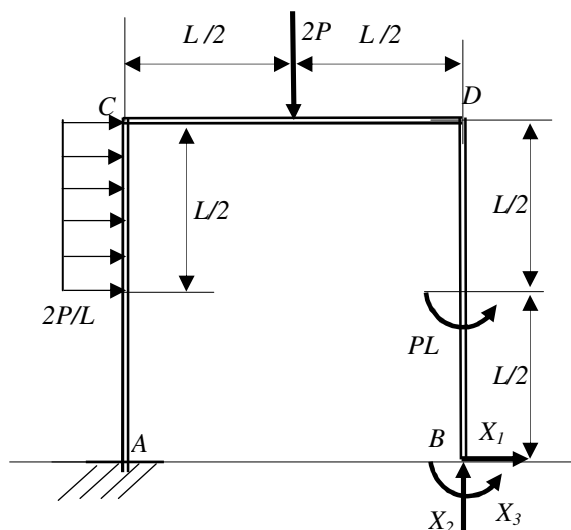


Fig. 2. The base system and the hyperstatic unknowns.

The canonical equations of the force method are:

$$\begin{cases} \delta_{10} + \delta_{11} \cdot X_1 + \delta_{12} \cdot X_2 + \delta_{13} \cdot X_3 = 0 \\ \delta_{20} + \delta_{21} \cdot X_1 + \delta_{22} \cdot X_2 + \delta_{23} \cdot X_3 = 0 \\ \delta_{30} + \delta_{31} \cdot X_1 + \delta_{32} \cdot X_2 + \delta_{33} \cdot X_3 = 0 \end{cases} \quad (1)$$

where:

$\delta_{10}$ ,  $\delta_{20}$  and  $\delta_{30}$  are the displacements produced by the given forces in the base system, on the direction of the hyperstatic efforts  $X_1$ ,  $X_2$  and  $X_3$ ;

$\delta_{ij}$  are the displacements produced by the unit forces  $X_j = 1$ ,  $j=1,2$  and unit bending moment  $X_3 = 1$  in the base system on the direction  $X_i$ .

The determination of the displacements  $\delta_{10}$ ,  $\delta_{20}$  and  $\delta_{30}$  and  $\delta_{11}$ ,  $\delta_{12}$ , ...  $\delta_{33}$  for the base system in Fig. 2 will be performed using the Mohr-Maxwell method, considering only the displacements produced by bending moments (**Error! Reference source not found., Error! Reference source not found.,Error! Reference source not found.**):

$$\begin{aligned} \delta_{i0} &= \sum \int \frac{m_i M^0}{EI} dx ; \\ \delta_{ik} &= \sum \int \frac{m_i m_k}{EI} dx \quad i, k = 1, 2, 3. \end{aligned} \quad (2)$$

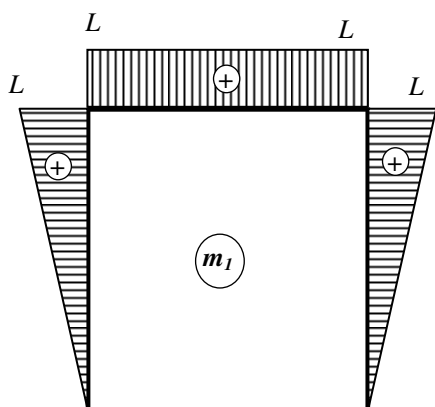


Fig. 3. The  $m_1$  diagram.

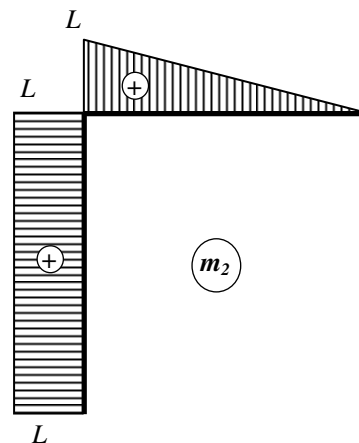


Fig. 4. The  $m_2$  diagram.

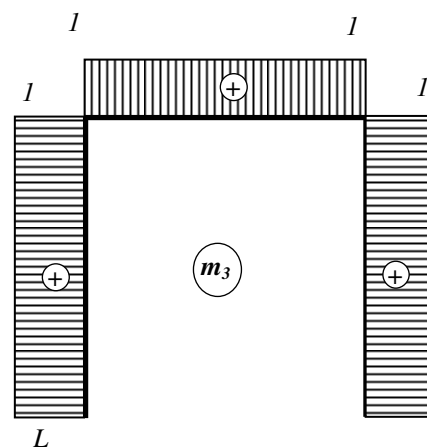


Fig. 5. The  $m_3$  diagram.

### 3. OBTAINED RESULTS

The displacements corresponding to the unit forces  $\delta_{ij}$  were determined using the equations (2) and the obtained values are presented in Table 2.

Table 2. Displacements corresponding to the unit forces.

$\delta_{11} = \frac{5}{3} \cdot \frac{L^3}{EI}$	$\delta_{22} = \frac{4}{3} \cdot \frac{L^3}{EI}$
$\delta_{12} = \delta_{21} = \frac{L^3}{EI}$	$\delta_{13} = \delta_{31} = 2 \cdot \frac{L^2}{EI}$
$\delta_{33} = 3 \cdot \frac{L}{EI}$	$\delta_{23} = \delta_{32} = \frac{3}{2} \cdot \frac{L^2}{EI}$

The parametric expressions of the displacements  $\delta_{10}$ ,  $\delta_{20}$  and  $\delta_{30}$  as function of  $\alpha$ ,  $\beta$ ,  $\eta$ ,  $\mu$ ,  $\omega$  and  $\theta$  are obtained for each of the six loads, which independently act on the base system, using the diagrams in 6 - 14. The loads were considered as:  $N_i = PL$ ;  $P_i = P$ ;  $q_{ij} = P/L$ .

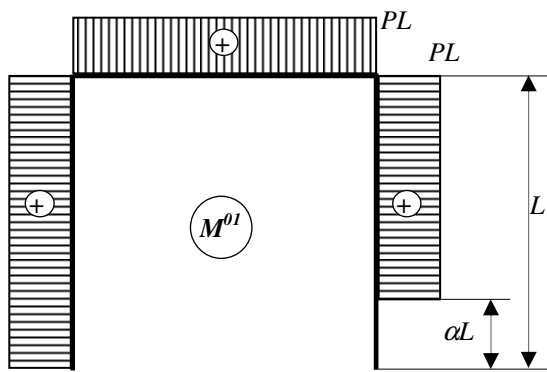


Fig. 4. Moment diagram for  $N_1$ .

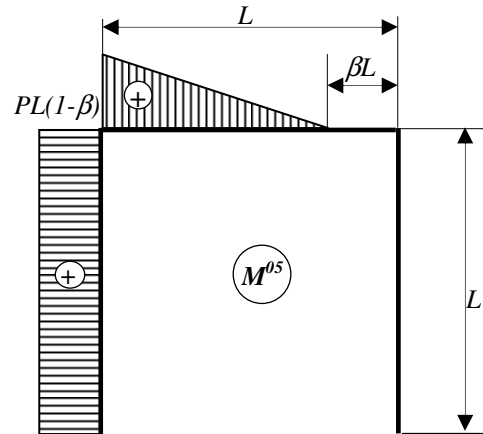


Fig. 10. Moment diagram for  $P_2$ .

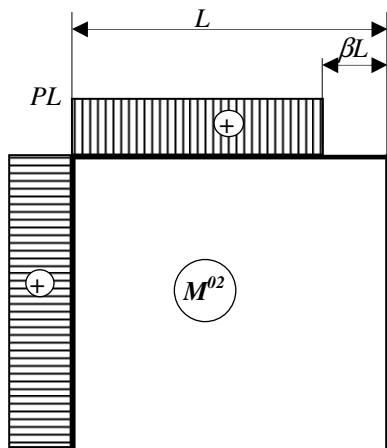


Fig. 5. Moment diagram for  $N_2$ .

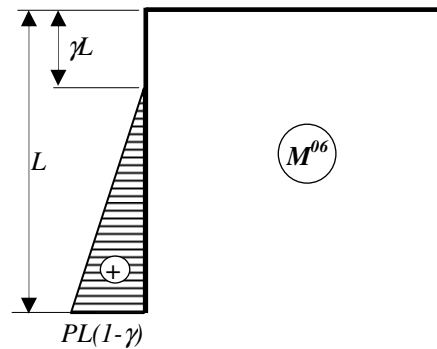


Fig. 11. Moment diagram for  $P_3$ .

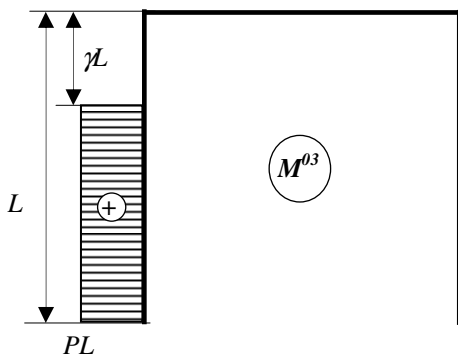


Fig. 8. Moment diagram for  $N_3$ .

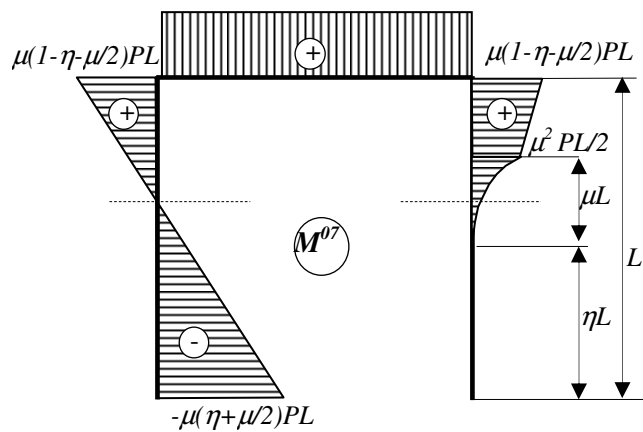


Fig. 12. Moment diagram for  $q_1$ .

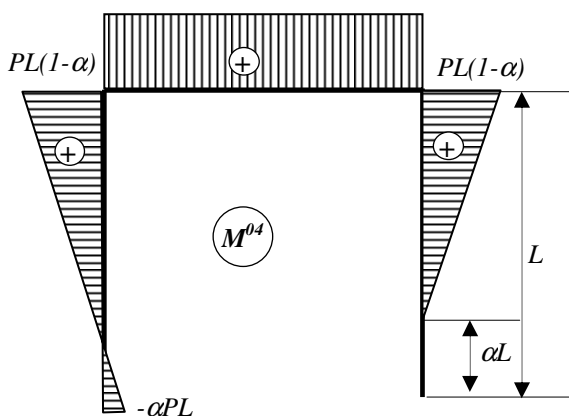


Fig. 9. Moment diagram for  $P_1$ .

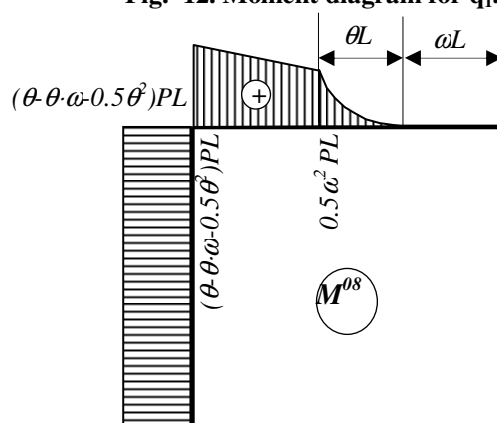


Fig. 13. Moment diagram for  $q_2$ .

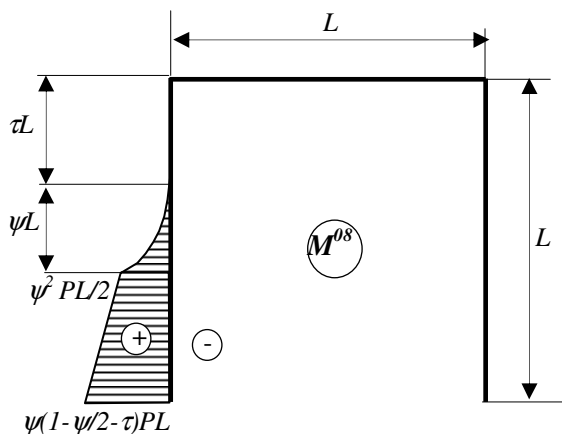


Fig. 14. Moment diagram for  $q_3$ .

The obtained parametric expressions of the displacements are summarized in **Error! Reference source not found.** - **Error! Reference source not found.**

Table 3. Parametric expressions of the displacements  $\delta_{10}$ .

Applied Loads		Parametric expressions for $\delta_{10}$
$N_1$	$\alpha$	$\delta_{10} = \frac{4 - \alpha^2}{2} \cdot \frac{PL^3}{EI}$
$N_2$	$\beta$	$\delta_{10} = \frac{3 - 2\beta}{2} \cdot \frac{PL^3}{EI}$
$N_3$	$\gamma$	$\delta_{10} = \frac{(1 - \gamma)^2}{2} \cdot \frac{PL^3}{EI}$
$P_1$	$\alpha$	$\delta_{10} = \frac{(1 - \alpha)(8 - \alpha - \alpha^2) - 3\alpha + 2}{6} \cdot \frac{PL^3}{EI}$
$P_2$	$\beta$	$\delta_{10} = \frac{(1 - \beta)(2 - \beta)}{2} \cdot \frac{PL^3}{EI}$
$P_3$	$\gamma$	$\delta_{10} = \frac{(1 - \gamma)^3}{6} \cdot \frac{PL^3}{EI}$
$q_1$	$\eta$ $\mu$	$\delta_{10} = \frac{0,25\mu^4 + \eta\mu^3 + (1,5\eta^2 - 6)\mu^2 + (\eta^3 - 12\eta + 10)\mu}{6} \cdot \frac{PL^3}{EI}$
$q_2$	$\omega$ $\theta$	$\delta_{10} = \frac{\theta^3 - 3(1,5 - \omega)\theta^2 + 3(\omega^2 - 3\omega + 2)\theta}{6} \cdot \frac{PL^3}{EI}$
$q_3$	$\tau$ $\psi$	$\delta_{10} = \frac{-0,25\psi^4 + (\tau - 1)\psi^3 - 1,5(1 - \tau)^2\psi^2 + (1 - \tau)^3\psi}{6} \cdot \frac{PL^3}{EI}$

Table 3. Parametric expressions of the displacements  $\delta_{20}$ .

Applied Loads		Parametric expressions for $\delta_{20}$
$N_1$	$\alpha$	$\delta_{20} = \frac{3}{2} \cdot \frac{PL^3}{EI}$
$N_2$	$\beta$	$\delta_{20} = \frac{3 - \beta^2}{2} \cdot \frac{PL^3}{EI}$

$N_3$	$\gamma$	$\delta_{20} = (1 - \gamma) \cdot \frac{PL^3}{EI}$
$P_1$	$\alpha$	$\delta_{20} = \frac{2 - 3\alpha}{2} \cdot \frac{PL^3}{EI}$
$P_2$	$\beta$	$\delta_{20} = \frac{(1 - \beta)(8 - \beta - \beta^2)}{6} \cdot \frac{PL^3}{EI}$
$P_3$	$\gamma$	$\delta_{20} = \frac{(1 - \gamma)^2}{2} \cdot \frac{PL^3}{EI}$
$q_1$	$\eta$ $\mu$	$\delta_{20} = \frac{\mu(4 - 3\mu - 6\eta)}{4} \cdot \frac{PL^3}{EI}$
$q_2$	$\omega$ $\theta$	$\delta_{20} = \frac{0,25\theta^4 + \omega\theta^3 + (1,5\omega^2 - 4,5)\theta^2 + (\omega^3 - 9\omega + 8)\theta}{6} \cdot \frac{PL^3}{EI}$
$q_3$	$\tau$ $\psi$	$\delta_{20} = \frac{\psi^3 - 3(1 - \tau)\psi^2 + 3(1 - \tau)^2\psi}{6} \cdot \frac{PL^3}{EI}$

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Table 5. Parametric expressions of the displacements  $\delta_{30}$ .

Applied Loads		Parametric expressions for $\delta_{30}$
$N_1$	$\alpha$	$\delta_{30} = (3 - \alpha) \cdot \frac{PL^2}{EI}$
$N_2$	$\beta$	$\delta_{30} = (2 - \beta) \cdot \frac{PL^2}{EI}$
$N_3$	$\gamma$	$\delta_{30} = (1 - \gamma) \cdot \frac{PL^2}{EI}$
$P_1$	$\alpha$	$\delta_{30} = \frac{\alpha^2 - 6\alpha + 4}{2} \cdot \frac{PL^2}{EI}$
$P_2$	$\beta$	$\delta_{30} = \frac{(1 - \beta)(3 - \beta)}{2} \cdot \frac{PL^2}{EI}$
$P_3$	$\gamma$	$\delta_{30} = \frac{(1 - \gamma)^2}{2} \cdot \frac{PL^2}{EI}$
$q_1$	$\eta$ $\mu$	$\delta_{30} = \frac{\mu^3 - 3(3 - \eta)\mu^2 + 3(\eta^2 - 6\eta + 4)\mu}{6} \cdot \frac{PL^2}{EI}$
$q_2$	$\omega$ $\theta$	$\delta_{30} = \frac{\theta^3 - 3(2 - \omega)\theta^2 + 3(\omega^2 - 4\omega + 3)\theta}{6} \cdot \frac{PL^2}{EI}$
	$\tau$ $\psi$	$\delta_{30} = \frac{\psi^3 - 3(1 - \tau)\psi^2 + 3(1 - \tau)^2\psi}{6} \cdot \frac{PL^2}{EI}$

In **Error! Reference source not found.**5 the hyperstatic frame configuration is represented for the case of the particular values from Table 1.

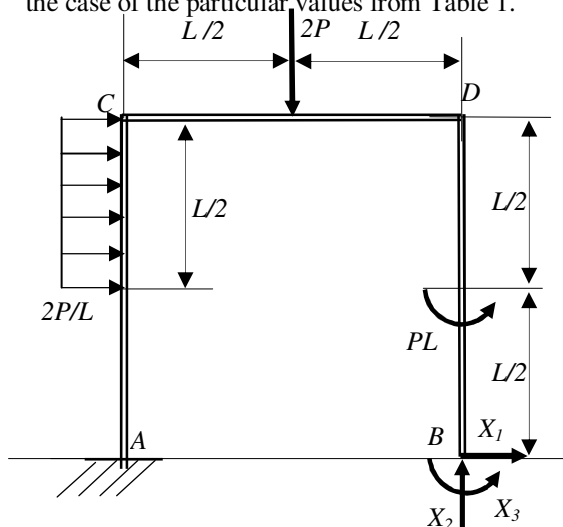


Fig. 5. The base system and the hyperstatic unknowns.

According to the relations in **Error! Reference source not found.** - **Error! Reference source not found.**, the displacements are:

$$\left\{ \begin{aligned} \delta_{10} &= \left[ \frac{4-\alpha^2}{2} \cdot 1 + \frac{(1-\beta)(2-\beta)}{2} \cdot (-2) + \frac{-0,25\gamma^4 + \gamma^3 - 1,5\gamma^2 + \gamma}{6} \cdot (-2) \right] \frac{PL^3}{EI} \\ \Rightarrow \delta_{10} &= \frac{67 PL^3}{64 EI} = 1,046875 \frac{PL^3}{EI} \end{aligned} \right. \quad (3)$$

$$\delta_{20} = \left[ \frac{3}{2} \cdot 1 + \frac{(1-\beta)(8-\beta-\beta^2)}{6} \cdot (-2) + \frac{\gamma^3 - 3\gamma^2 + 3\gamma}{6} \cdot (-2) \right] \frac{PL^3}{EI} = 0 \quad (4)$$

$$\delta_{30} = \left[ (3-\alpha) \cdot 1 + \frac{(1-\beta)(3-\beta)}{6} \cdot (-2) + \frac{\gamma^3 - 3\gamma^2 + 3\gamma}{6} \cdot (-2) \right] \frac{PL^2}{EI}$$

$$\Rightarrow \delta_{30} = \frac{23 PL^2}{24 EI} = 0,958333 \frac{PL^2}{EI} \quad (5)$$

Replacing the obtained values in the equations system (1), the hyperstatic unknowns may be now determined:

$$\left\{ \begin{aligned} \frac{5}{3} \cdot \frac{L^3}{EI} \cdot X_1 + \frac{L^3}{EI} \cdot X_2 + 2 \cdot \frac{L^2}{EI} \cdot X_3 &= -\frac{67 PL^3}{64 EI} \\ \frac{L^3}{EI} \cdot X_1 + \frac{4}{3} \cdot \frac{L^3}{EI} \cdot X_2 + \frac{3}{2} \cdot \frac{L^2}{EI} \cdot X_3 &= 0 \\ 2 \cdot \frac{L^2}{EI} \cdot X_1 + \frac{3}{2} \cdot \frac{L^2}{EI} \cdot X_2 + 3 \cdot \frac{L}{EI} \cdot X_3 &= -\frac{23 PL^2}{24 EI} \end{aligned} \right. \quad (6)$$

$$\left\{ \begin{aligned} X_1 &= -\frac{235}{192} P = -1,223958 P \\ X_2 &= \frac{23}{28} P = 0,821429 P \\ X_3 &= \frac{173}{2016} P = 0,085813 P \end{aligned} \right. \quad (7)$$

## CONCLUSIONS

- Using MATHCAD 14, any hyperstatic systems similar to the presented case can be analyzed, by simply modifying the parameters given in the relations above.
- The parametric relations can also be used for the verification of the results obtained for this type of systems by means of other numerical analysis methods.

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